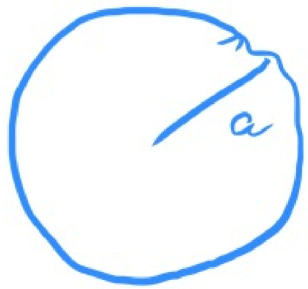
Last class: Laplace's Equation for a Disk



$\nabla^2 u = 0$

u prescribed at boundary

in polar coordinates

$$\nabla^2 u = \frac{1}{r}\partial/_{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u(a,\theta) = f(\theta)$$

separate variables: $u(r,\theta) = G(r)\,\phi(\theta)$

got 2 ODE's (a) $\phi'' = -\lambda\phi$

(b) $r^2 G'' + rG' = \lambda G$

boundary cond. $\phi(-\pi) = \phi(\pi)$

$$\phi'(-\pi) = \phi'(\pi)$$

$\Rightarrow$ get eigenvalues $\lambda = n^2$, $n = 0, 1, 2 \ldots$

and eigenfunctions $\boxed{\phi(\theta) = \begin{cases} \text{const} & n = 0 \\ \cos n\theta, \sin n\theta & n > 0 \end{cases}}$

solutions of

$$r^2 G'' + r G' = n^2 G:$$

$\underline{n=0}$: $G = \text{const.}$ or $G = \log r$

$n > 0$ $G = r^n$ or $G = r^{-n}$

ignore
solutions should
be bounded for $r=0$

To calculate B_m, we take brackets $(\cdot, \cos m\theta)$ of equation encircled in green

$$(f, \cos m\theta) = \sum_{n=0}^{\infty} A_n a^n (\cos n\theta, \cos m\theta) + \sum_{n=1}^{\infty} B_n a^n (\sin n\theta, \cos m\theta)$$

by orthogonality

$$\int_{-\pi}^{\pi} \cos^2 n\theta \, d\pi = \begin{cases} \pi & n \neq 0 \\ 2\pi & n = 0 \end{cases}$$

$$(\sin n\theta, \cos m\theta) = 0 \qquad n \neq 0$$

$$(\cos n\theta, \cos m\theta) = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

$$(\underbrace{\cos 0\theta}_{=1}, \underbrace{\cos 0\theta}_{=1}) = (1, 1) = 2\pi$$

yellow equ. $\Rightarrow$

solve for B_m $\rightarrow$

$$\int_{-\pi}^{\pi} f(\theta) \cos m\theta \, d\theta = B_m a^m \int \cos^2 m\theta \, d\theta$$

General solution

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta}$$

const. sol. for $n=0$

Calculate A_n's and B_n's from boundary condition

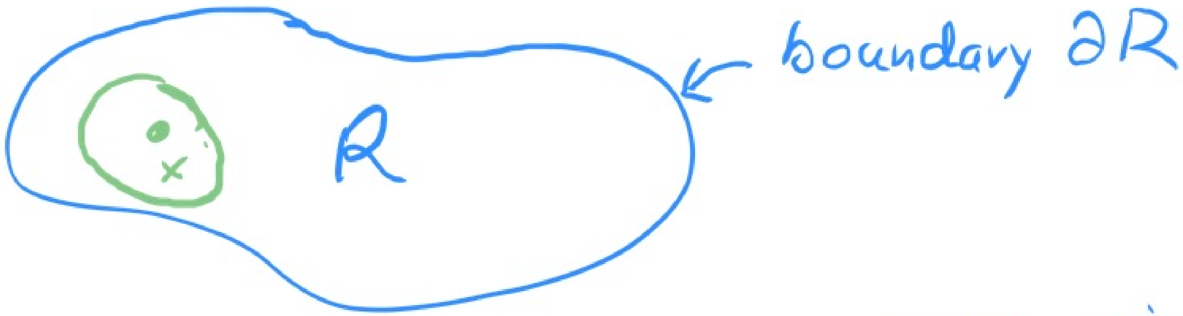
$$\boxed{f(\theta) = u(a,\theta) = \sum_{n=0}^{\infty} \underbrace{A_n a^n} \cos n\theta + \sum_{n=1}^{\infty} \underbrace{B_n a^n} \sin n\theta}$$

$$\Rightarrow \quad A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = \text{average temperature on boundary circle}$$

$$A_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta$$

$$B_n a^n = \quad \cdots \quad \sin n\theta \quad \ldots$$

2.5.4 Qualitative Properties of Laplace's Equation



Problem: $\nabla^2 u = 0$ and $u(\vec{x}) = f(\vec{x})$ given
for $\vec{x}$ in ∂R

(1) Mean Value Theorem

If R = dish of radius a

$\Rightarrow u(0,\theta) = A_0 + \sum r^n \ldots$ ($r=0$)

center of dish $= A_0$

$=$ average over boundary temp.

same holds for green circle within R

i.e. $\boxed{u(\vec{x}) = \text{average over green circle around } \vec{x}}$

Mean Value Theorem.

2. Maximum Principle

$\boxed{\text{If } \nabla^2 u = 0 \text{ on } R \text{ then } u \text{ reaches its max. value on boundary of } R}$

proof by contradiction

assume u reaches max at $\vec{x}_0$

x_0 not in ∂R

$\Rightarrow u(\vec{x}_0) =$ average over green circle

$\frac{1}{2\pi}\int u(\vec{x})\, dt \leftarrow$ integral over green circle

$$u(\vec{x}) \leq u(\vec{x_0})$$

$$\Rightarrow \int_C u(\vec{x}) \leq u(\vec{x_0})$$

$$\parallel$$

$u(\vec{x_0})$ by mean value theorem

$\Rightarrow$ $u(\vec{x}) = u(\vec{x_0})$ on green circle.

true for any circle around $\vec{x_0}$

$\Rightarrow$ u must be constant.

<u>Conclusion:</u> only possible counter example would be constant function

✓

③ Wellposedness and Uniqueness of solutions

uniqueness: Assume u_1 and u_2 are two solutions of

$$\boxed{\nabla^2 u = 0 \text{ on } R \quad \text{and} \quad u(\vec{x}) = f(\vec{x}) \quad \vec{x} \text{ on } \partial R}$$

<u>claim:</u> $u_1 = u_2$, i.e we only have one solution of

proof. Let $g = u_1 - u_2$

$\Rightarrow \nabla^2 g = 0$

and $g(\vec{x}) = u_1(\vec{x}) - u_2(\vec{x}) = f(\vec{x}) - f(\vec{x}) = 0$ for $\vec{x} \in \partial R$

$\Rightarrow$ max of g on $R \underset{\text{max. principle}}{=}$ max of g on $\partial R = 0$

$$\boxed{u_1(x) = u_2(x) = f(\vec{x}) \text{ for } \vec{x} \text{ in } \partial R}$$

$\Rightarrow \quad \underset{u_1(x)-u_2(x)}{\underset{\|}{g(x)}} \leq 0 \quad$ for all x

$\Rightarrow \boxed{u_1(x) \leq u_2(x)} \quad \forall x$ in $\mathbb{R}$

by same argument for $h(x) = u_2(x) - u_1(x)$

$\Rightarrow \quad \max h(x) = 0$

$\Rightarrow \quad \boxed{u_2(x) \leq u_1(x)} \quad \forall x$ in $\mathbb{R}$

Conclusion:

$u_1(x) = u_2(x)$ for all x in $\mathbb{R}$

<u>well posedness</u>: If we vary initial conditions slightly

$\Rightarrow$ solutions only vary slightly.